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INTRODUCTION TO SYMBOLIC LOGIC

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To Carrot, Klunky, Imbrecilla,
and Question Mark

Thus formal analyticity is another way to talk about sound argument forms. *An argument form is sound if, and only if, its corresponding conditional is formally analytic.* Thus, the argument form

All A are B.
All B are C.

Therefore, all A are C.

is sound, because the corresponding conditional "If all A are B and all B are C, then all A are C" is formally analytic—regardless of what we substitute for A, B, and C, we get an analytic statement. This, then, is the most important connection between the two conceptions of logic. It gives us a way of talking about sound argument forms in terms of the formal analyticity of statements.

II THE PROPOSITIONAL CALCULUS

II.1 THE SENTENTIAL CONNECTIVES

There is a large class of statements and arguments whose forms can be expressed adequately using the expressions "and," "or," "if . . . then," "if and only if," and "it is not the case that." These expressions are called the *sentential connectives*, because they are used to connect sentences to form larger sentences. For example, the statement "Either it is raining or it is not the case that it is raining" has the form "P or it is not the case that P," where now P stands for "It is raining." Similarly, the statement "If John goes to the party, then Joe will stay home" has the form "If P then Q," where P stands for "John will go to the party" and Q stands for "Joe will stay home."

Unfortunately, in natural languages, such as English, statements that have the same form do not always look the same. For example, the statements "Either it is snowing or it is not the case that it is snowing" and "Either it is sleeting or it is not sleeting" have the same form; they are both formed by saying that either one statement is true or else a second statement is true, where the second statement simply denies the truth of the first statement. But they do not look quite the same because the "not" is placed differently in the two statements. It is therefore advantageous to introduce an artificial symbolism to represent the form of a statement. We replace "not" by "$\sim$," and then both "It is not the case that it is sleeting" and "It is not sleeting" can be represented by "$\sim$ (it is sleeting)."

Letting P and Q be arbitrary statements, we can introduce the following symbols:

Symbol	Meaning
$\sim P$	It is not the case that P.
$(P \,\&\, Q)$	P and Q (or, both P and Q).
$(P \vee Q)$	P or Q (or, either P or Q).
$(P \supset Q)$	If P then Q.
$(P \equiv Q)$	P if and only if Q.

These symbols are abbreviations for the sentential connectives.[1] Statements formed using these symbols have names, as follows: $\sim P$ is the *negation* of P; $(P \,\&\, Q)$ is the *conjunction* of P and Q, and P and Q are the *conjuncts*; $(P \vee Q)$ is the *disjunction* of P and Q, and P and Q are the *disjuncts*; $(P \supset Q)$ is a *conditional*, and P is the *antecedent*, and Q is the *consequent*; $(P \equiv Q)$ is a *biconditional*.

Using these symbols we can represent the forms of quite complex statements. We have been using capital letters to symbolize the simple statements that form the parts of compound statements. We might occasionally need more than 26 letters, so we can also use capital letters with subscripts, such as P_7 or Q_{13}. It should be emphasized that the same letter with different subscripts, such as P and P_7, can be used to represent totally unrelated statements. The fact that P occurs in P_7 is just a coincidence. Capital letters, with or without subscripts, used this way are called *sentential letters*. In the statement "Either it is raining or it is not the case that it is raining," we can let P_3 be the statement "It is raining." Then the above statement becomes "Either P_3 or it is not the case that P_3." Using the symbols for the sentential connectives, this in turn can be symbolized as "Either P_3 or $\sim P_3$," and then as $(P_3 \vee \sim P_3)$.

Now consider some more examples of symbolizing the forms of statements like "If he comes we will have the party at his house, and if he doesn't come then we will have the party at Jones' house." Letting P be the statement "He comes," Q the statement "We will have the party at his house," and R the statement "We will have the party at Jones' house," we can first symbolize the statement as "If P then Q, and if it is not the case that P then R." This in turn can be symbolized as "If P then Q, and if $\sim P$ then R," and then as "$(P \supset Q)$ and $(\sim P \supset R)$," and finally as $[(P \supset Q) \,\&\, (\sim P \supset R)]$.

[1] In what follows, these symbols will be called the sentential connectives, although strictly speaking they are abbreviations for the sentential connectives. The sentential connectives themselves are English words and phrases.

As another example, consider "If Jones needs money, then either he will reduce prices or he will apply to the bank for a loan." Letting P be "Jones needs money," Q be "He will reduce prices," and R be "He will apply to the bank for a loan," this statement can be symbolized, first as "If P then either Q or R," then as "If P then $(Q \vee R)$," and finally as $[P \supset (Q \vee R)]$.

There are two things that should be noticed about the preceding examples. The first is that in symbolizing relatively complex statements like "If he comes we will have the party at his house, and if he doesn't come then we will have the party at Jones' house," we begin by symbolizing the smallest parts, and then we construct the successively larger parts one at a time until we finally get the whole statement. The symbolization went as follows:

$$\underbrace{\text{If } P \text{ then } Q}\text{, and } \underbrace{\text{if it is not the case that } P \text{ then } R}$$

$$(P \supset Q) \text{ and } \underbrace{\text{if} \quad \sim P \quad \text{then } R}$$

$$\underbrace{(P \supset Q) \text{ and } \quad (\sim P \supset R)}$$

$$[(P \supset Q) \,\&\, (\sim P \supset R)]$$

It is always best to begin by symbolizing the smallest parts of a statement first, and then constructing the symbolization of the successively larger parts one at a time in terms of the parts that make them up. It is unwise to try to symbolize an entire statement in one fell swoop if the statement is at all complicated.

Second, notice the use of parentheses. Each time we symbolize a part of the statement—unless that part is a negation—we enclose it in parentheses to keep it separate from the other parts of the statement. The parentheses take the place of commas and other grammatical conventions of English. We must always be careful to put the parentheses in, or the resulting symbolization will be ambiguous. Consider the two statements "It is not true that Jones has a girlfriend and his wife is going to divorce him" and "It is not true that Jones has a girlfriend, and his wife is going to divorce him." These obviously mean quite different things. The first denies that it is true both that Jones has a girlfriend and that his wife is going to divorce him, whereas the second says that Jones does not have a girlfriend, but his

wife is going to divorce him anyway. These two statements would be symbolized as $\sim(P \& Q)$ and $(\sim P \& Q)$ respectively. But if we omitted the parentheses in symbolizing these statements and just wrote $\sim P \& Q$, we would not know which of these two different statements were meant.

It is not necessary to enclose a negation (a statement of the form $\sim P$) in parentheses, because "$\sim$" does not really connect sentences—it acts on a single sentence. But for any of the other sentential connectives it is necessary to use parentheses each time the connective is used.

II.1 EXERCISES

Symbolize the forms of the following statements:

1. If it rains today the ground will be wet, and we will not be able to have a picnic.
2. It is not true that if it is cloudy then it will rain.
3. If the sun shines in the morning it will rain, and if the sun does not shine in the morning then it will not rain.
4. If we get plenty of sunshine, then if it rains the flowers will grow.
5. It is not the case that, there is a woman in the next room if and only if Jim said there is.
6. It is not the case that there is a woman in the next room, if and only if, Jim said there is.
7. Either the entrails will contain cockroaches or they will not, and if they do then the gods are angered, and if they do not then Venus will be in apposition to Jupiter.
8. If she is an acrobat or she is a clown, then she lives in that trailer.
9. Harry will go to the bank today if, and only if, the market drops, and if Harry goes to the bank today the supervisors will hide, and if the supervisors hide and Harry does not go to the bank today, then Emmett will lose his shirt on the stock market.
10. If Francis Bacon wrote *Hamlet* and Shakespeare wrote *Macbeth*, then either Shakespeare was Bacon, or the theater manager was a crook.

II.2 FORMULAS OF THE PROPOSITIONAL CALCULUS

Using sentential letters to stand for statements, and then combining them with sentential connectives, we can construct quite complex statement forms, called *formulas of the propositional calculus*. We can give

precise rules for constructing formulas of the propositional calculus. The simplest formulas are simply sentential letters, with or without subscripts, such as P, Q, P_{13}, R_{129}, and so forth. Let us call these *atomic formulas of the propositional calculus* because they are the atoms from which more complicated formulas are constructed. Then all other formulas of the propositional calculus can be constructed by successive applications of the following rules:

1. An atomic formula is a formula.
2. If P is any formula, then $\sim P$ is a formula.
3. If P and Q are any formulas, then $(P \& Q)$ is a formula.
4. If P and Q are any formulas, then $(P \vee Q)$ is a formula.
5. If P and Q are any formulas, then $(P \supset Q)$ is a formula.
6. If P and Q are any formulas, then $(P \equiv Q)$ is a formula.

All formulas of the propositional calculus can be constructed by repeated application of these rules. Consider the formula

$$(P \supset (\sim Q \& R))$$

We can build this formula up using the above six rules as follows. We begin with the smallest parts and work outwards. By Rule 1, P, Q, and R are formulas. Then by Rule 2, $\sim Q$ is a formula. By Rule 3, as $\sim Q$ and R are both formulas, $(\sim Q \& R)$ is a formula. Then by Rule 5, as P and $(\sim Q \& R)$ are both formulas, $(P \supset (\sim Q \& R))$ is a formula.

We can construct very complicated formulas using these rules. For example, consider the formula

$$((P \supset (Q_3 \equiv \sim R_4)) \\ \equiv \sim\sim(\sim P \& \sim(Q_3 \equiv (R_4 \vee (R_{17} \& \sim\sim P)))))$$

Let us see how we would build this formula up using the six rules. Again, we begin with the smallest parts and work outwards. By Rule 1, P, Q_3, R_4, and R_{17} are formulas, because they are atomic formulas. As P and R_4 are formulas, we can use Rule 2 to construct the formulas $\sim P$ and $\sim R_4$. As $\sim P$ is then a formula, by Rule 2 again, $\sim\sim P$ is a formula. Then as R_{17} and $\sim\sim P$ are both formulas, by Rule 3, $(R_{17} \& \sim\sim P)$ is a formula. Then as R_4 and $(R_{17} \& \sim\sim P)$ are both formulas, by Rule 4, $(R_4 \vee (R_{17} \& \sim\sim P))$ is a formula. Then as Q_3 and $(R_4 \vee (R_{17} \& \sim\sim P))$ are both formulas, by Rule 6,

$$(Q_3 \equiv (R_4 \vee (R_{17} \& \sim\sim P)))$$

is a formula. Then by Rule 2,

$$\sim(Q_3 \equiv (R_4 \vee (R_{17} \& \sim\sim P)))$$

is a formula. We have already seen that $\sim P$ is a formula, so by Rule 3,

$$(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$$

is a formula. Then by Rule 2,

$$\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$$

is a formula, and then by Rule 2 again,

$$\sim\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$$

is a formula. This gives us the right side of the biconditional. Now working on the other side, as Q_3 and $\sim R_4$ are both formulas, by Rule 6, $(Q_3 \equiv \sim R_4)$ is a formula. P is a formula, so by Rule 5, $(P \supset (Q_3 \equiv \sim R_4))$ is a formula. Both sides of the biconditional have now been constructed, and then by Rule 6,

$$((P \supset (Q_3 \equiv \sim R_4)) \equiv \sim\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P)))))$$

is a formula. We can diagram the construction of this formula as follows:

$P \quad Q_3 \quad R_4 \quad P \quad Q_3 \quad R_4 \quad R_{17} \quad P$	Rule 1
$\sim R_4 \quad \sim P \quad \sim P$	Rule 2
$\sim\sim P$	Rule 2
$(R_{17} \mathbin{\&} \sim\sim P)$	Rule 3
$(R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))$	Rule 4
$(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P)))$	Rule 6
$\sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P)))$	Rule 2
$(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$	Rule 3
$\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$	Rule 2
$\sim\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P))))$	Rule 2
$(Q_3 \equiv \sim R_4)$	Rule 6
$(P \supset (Q_3 \equiv \sim R_4))$	Rule 5
$((P \supset (Q_3 \equiv \sim R_4)) \equiv \sim\sim(\sim P \mathbin{\&} \sim(Q_3 \equiv (R_4 \vee (R_{17} \mathbin{\&} \sim\sim P)))))$	Rule 6

Notice that whenever we construct a conjunction, disjunction, conditional, or biconditional, we enclose it in parentheses. In order to make formulas easier to read, parentheses are often replaced by brackets. Thus we might write the above formula as

$$\{[P \supset (Q_3 \equiv \sim R_4)] \equiv \sim\sim[\sim P \mathbin{\&} \sim(Q_3 \equiv \{R_4 \vee (R_{17} \mathbin{\&} \sim\sim P)\})]\}$$

By doing this we can tell at a glance which left parenthesis or bracket goes with which right parenthesis or bracket; thus, we can see more easily what the formula means.

Atomic formulas are the simplest formulas. Let us say that a formula

is *molecular* if it is not atomic. Then molecular formulas are those formulas that contain sentential connectives.

Sometimes we will want to talk about one formula being a *part of* another formula. This just means that the first occurs somewhere in the second. For example, P, Q, R, $(P \supset Q)$, $\sim(P \supset Q)$, and

$$[\sim(P \supset Q) \vee R]$$

are all parts of the formula $\sim[\sim(P \supset Q) \vee R]$. Let us also adopt the convention of saying that a formula is part of itself. Then

$$\sim[\sim(P \supset Q) \vee R]$$

is also a part of $\sim[\sim(P \supset Q) \vee R]$. The *atomic parts* of a formula are those parts of the formula that are atomic. Thus P, Q, and R are the atomic parts of $\sim[\sim(P \supset Q) \vee R]$, while $(P \supset Q)$, $\sim(P \supset Q)$, $[\sim(P \supset Q) \vee R]$ and $\sim[\sim(P \supset Q) \vee R]$ are parts that are not atomic.

II.2 EXERCISES

A. Show how the following formulas of the propositional calculus can be constructed using Rules 1 through 6. Diagram the construction as was done on page 14.

1. $((P \supset Q) \equiv (\sim(Q \supset P) \vee \sim\sim(Q \mathbin{\&} \sim P)))$
2. $\sim((Q \mathbin{\&} R) \supset \sim(\sim Q \mathbin{\&} (R \supset (Q \vee \sim R))))$
3. $((P \vee (Q \mathbin{\&} R)) \equiv \sim(\sim P \mathbin{\&} (\sim Q \vee \sim R)))$

B. Construct a table for each of the following expressions indicating: (1) whether it is a formula of the propositional calculus; (2) if it is a formula, whether it is atomic or molecular; (3) if it is a molecular formula, whether it is a negation, conjunction, disjunction, conditional or biconditional; (4) if it is a formula, what its atomic parts are; (5) if it is a formula, what its parts are:

1. P
2. $\sim\sim P$
3. $P \vee Q$
4. $((P \vee Q) \supset \sim Q)$
5. $(\sim(P \equiv Q))$
6. $(P \vee Q \supset R)$
7. $\sim((Q \mathbin{\&} R) \equiv (\sim Q \vee \sim R))$
8. $(P \vee P)$
9. $(P \mathbin{\&} \sim P)$
10. $(P \supset Q) \equiv \sim(Q \supset P)$

II.3 PARAPHRASING

Sometimes it is necessary to paraphrase a statement before it can be symbolized. Consider the statement "John and Joe both came to the party." We want to symbolize this as a conjunction, but it cannot be symbolized directly because "and" stands between two names rather than between two sentences. Before we can symbolize it we must paraphrase it so that the sentential connective connects two sentences, "John came to the party and Joe came to the party." Then letting P be "John came to the party" and Q be "Joe came to the party," we can symbolize it as $(P \& Q)$.

Another example of such paraphrasing is found in the following statement: "If either Kennedy or Khrushchev had been weaker willed concerning either Berlin or Cuba, the cold war would have turned into a hot war." This must be paraphrased first as "If either Kennedy had been weaker willed concerning either Berlin or Cuba, or Khrushchev had been weaker willed concerning either Berlin or Cuba, then the cold war would have turned into a hot war." Then this must be paraphrased again as "If either Kennedy had been weaker willed concerning Berlin or Kennedy had been weaker willed concerning Cuba, or Khrushchev had been weaker willed concerning Berlin or Khrushchev had been weaker willed concerning Cuba, then the cold war would have turned into a hot war." Finally then, this can be symbolized as $\{[(P \vee Q) \vee (R \vee S)] \supset T\}$.

Other kinds of paraphrasing may also be necessary. There are expressions in English such as "unless," "but," "if," "only if," "neither . . . nor," that are much like the sentential connectives. Whenever these occur in a statement, the statement must be paraphrased to replace them by sentential connectives. For example, "Neither Joe came to the party nor John came to the party" means the same thing as "Joe didn't come to the party and John didn't come to the party," and so it must be paraphrased in that way. In general, "Neither P nor Q" can be paraphrased as $(\sim P \& \sim Q)$.

"But" is like a forceful "and." We may say "He came but he didn't like it" rather than "He came and he didn't like it" merely to emphasize the second conjunct. This emphasis makes no difference for logic, so we can symbolize "P but Q" as simply $(P \& Q)$.

The behavior of the expressions "if" and "only if" is somewhat surprising. There seems to be a strong temptation to identify "P if Q" with "If P then Q," but in fact it should be the other way around; that is, "P if Q" means the same thing as "If Q then P." Consider the statement "The crops will be destroyed if there is a flood." To say this is not to say that the crops might not be destroyed anyway, for example, by a drought. But the statement "If the crops are destroyed there is a flood" precludes the possibility of them being destroyed by a drought, therefore it cannot be a proper paraphrase of "The crops will be destroyed if there is a flood." The proper paraphrase is "If there is a flood then the crops will be destroyed." In general, "P if Q" can be symbolized as $(Q \supset P)$.

"P only if Q" works just the other way around. It means "If P then Q." Consider the statement "The crops will be destroyed only if there is a flood." This means that the only way the crops can be destroyed is by a flood, and hence if the crops *are* destroyed then there must have been a flood. This then means "If the crops are destroyed then there is a flood."

Note that "P if and only if Q" is just the conjunction of "P if Q" and "P only if Q," and thus that $(P \equiv Q)$ is equivalent to

$$[(P \supset Q) \& (Q \supset P)].$$

One further expression that can be paraphrased in terms of the sentential connectives is "unless." "P unless Q" can be paraphrased as $(\sim Q \supset P)$. Suppose we want to paraphrase the statement "We will go to the beach unless it rains." This is the same thing as saying "If it doesn't rain then we will go to the beach;" that is $(\sim Q \supset P)$.

II.3 EXERCISES

Symbolize the forms of the following statements:

1. Neither Jack nor Jim will come unless Mary comes.
2. We will not get there on time unless we speed, but if we speed we might not get there at all.
3. We can get the door open only if we use an acetylene torch on it, but then the door will be ruined.
4. The river will not overflow its banks unless we either have an early thaw or heavy rains, but we will not have heavy rains.
5. Unless we have a flat tire, we can get there on time if we speed, but we will have a flat tire if we speed.

6. Neither Jack nor Jim will come if Mary comes, unless Joan and Mary both come.
7. Jeremy will get a Mercedes Benz for Christmas only if he does not offend Santa Claus, but Jeremy will offend Santa Claus if he does not believe in him, and Jeremy does not believe in Santa Claus.
8. Rain is imminent.
9. John will not come unless Jim comes, and Jim will not come if Jeffrey comes, but Jeffrey will only come if John does not come.
10. It will rain if the barometer drops, but if it rains it will cool off later, and it will not cool off later.

II.4 DIVERGENT USES

One thing to beware of in symbolizing the forms of statements is that there are divergent uses of some of the sentential connectives in which they do not have their ordinary meaning. For example, "and" sometimes means "and then" rather than simply "and." In its ordinary use, "P and Q" means "It is true that P, and it is true that Q." On this reading, "P and Q" means the same thing as "Q and P." For example, "This is red and that is white" means the same thing as "That is white and this is red." But consider the use of "and" in the following sentence: "She got married and had a baby." This clearly does not mean the same thing as "She had a baby and got married." The former means "She got married and then had a baby." This use of "and" cannot be symbolized simply as "&." The same thing is true of "and" in "He lay down and fell asleep."

The sentential connective that has the greatest number of divergent uses is "if... then." What will be called the standard use of "if... then" is that in which "If P then Q" is true under the same conditions as "Either $\sim P$ or Q"; that is, $(\sim P \vee Q)$. It is only this meaning of "If P then Q" that can be symbolized as $(P \supset Q)$. First it should be seen that "if... then" is at least sometimes used in this way. Consider the statement "If it rains then the crops will grow." Let us symbolize this as "If P then Q," where P means "It will rain" and Q means "The crops will grow." Now it will be argued that this is true under exactly the same circumstances as "Either it won't rain, or the crops will grow"; that is, $(\sim P \vee Q)$:

1. Suppose that "If P then Q" is true. So if it rains (if P is true) then Q is true. If it does not rain, then $\sim P$ is true. But either it will not rain or it will rain. So either it will not rain, and then $\sim P$ will be true, or else it will rain, and then Q will be true. Thus, no matter what happens, either $\sim P$ will be true or Q will be true; that is, $(\sim P \vee Q)$ will be true. So we see that if the statement "If P then Q" is true, then $(\sim P \vee Q)$ is true.

2. In order to show that "If P then Q" and $(\sim P \vee Q)$ are true under exactly the same circumstances, the converse of the above must now be proven, namely, that if $(\sim P \vee Q)$ is true then "If P then Q" is true. So let us suppose that $(\sim P \vee Q)$ is true. A disjunction "A or B" is true if, and only if, at least one of its disjuncts is true; that is, if, and only if, either A is true or B is true. Thus if $(\sim P \vee Q)$ is true, then either $\sim P$ is true or Q is true. Now it will be shown that "If P then Q" is true. If P is true, then $\sim P$ cannot be true. But either $\sim P$ or Q must be true. So if P is true then Q must be true; that is, "If P then Q" is true. Thus it has been shown that if $(\sim P \vee Q)$ is true, then "If P then Q" is true.

Hence the two statements, "If it rains the crops will grow," and "Either it won't rain or the crops will grow," are true in exactly the same circumstances. In this context "if... then" has what is called its "standard use."

However, there are other uses of "if... then" in which "if P then Q" is clearly not the same as "Either $\sim P$ or Q." One place this occurs is in what are called *counterfactual conditionals*. These are statements such as "If this match had been struck it would have lit," which tell us that if something that did not happen *had* happened, then something else would have been the case. To see that these conditionals cannot be translated into disjunctions, let us suppose that we have a match which was not struck. Then exactly one of the following two statements is true: "If this match had been struck it would have lit"; "If this match had been struck it wouldn't have lit." If "if... then" had its standard use in these statements, they would mean "Either this match wasn't struck or else it lit" and "Either this match wasn't struck or else it didn't light" respectively. But since the match was not struck, *both* of these disjunctions are true (because "This match wasn't struck" is true). Therefore, these disjunctions cannot mean the same thing as the counterfactual conditionals, because it is impossible for both of the counterfactual conditionals to be true at the same time. Either "If this match had been struck it would have lit" or "If this match had been struck it wouldn't have lit" is true, but they cannot both be true.